

# A SIMPLE CONSTRUCTION OF GENERALIZED COMPLEX MANIFOLDS

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## ABSTRACT

We construct a natural generalized complex structure on the total space of any bundle endowed with a Chern connection and whose typical fibre is a homogeneous symplectic manifold. This extends constructions of [1] and [2] and leads to natural examples of holomorphic maps between generalized complex manifolds.

## INTRODUCTION

The notion of Dirac structure is the natural generalization of the notions of symplectic and Poisson structures. This is achieved through the essentially equivalent notion of presymplectic foliation and, up to the integrability, it corresponds, on any manifold  $M$ , to a subbundle of  $TM \oplus T^*M$  which is maximally isotropic with respect to the inner product induced by the natural identification of the tangent bundle with its bidual.

On complexifying, one is led to maximal isotropic subbundles of  $T^{\mathbb{C}}M \oplus (T^{\mathbb{C}}M)^*$ . Obviously, any such subbundle  $L$ , which further satisfies  $L \cap \overline{L} = 0$ , is the  $i$ -eigenbundle of an orthogonal complex structure on  $TM \oplus T^*M$ . Together with the suitable integrability condition, this gives the notion of *generalized complex structure* (see [3]).

Now, up to products and  $B$ -field transformations, the following seem to exhaust the known ‘God given’ (that is, not manufactured) examples of such structures:

- (i) the (classical) complex structures;
- (ii) the symplectic structures;
- (iii) the generalized complex structures associated to the holomorphic Poisson structures;
- (iv) the generalized complex structures associated to the holomorphic foliations on Kähler manifolds.

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The aim of this note is to add one more natural construction to this list (Theorem 1.1; see, also, Corollary 1.2). This extends constructions of [1] and [2] and, also, leads to natural examples of holomorphic maps between generalized complex manifolds.

Our construction works for any bundle endowed with a Chern connection and whose typical fibre is a homogeneous symplectic manifold, and seems to, also, give examples of ‘generalized holomorphic bundles’ whose typical fibres are symplectic.

## 1. THE CONSTRUCTION

Let  $(P, M, G^{\mathbb{C}})$  be a holomorphic principal bundle whose structural group is the complexification of a Lie group  $G$ . Suppose that  $(P, M, G^{\mathbb{C}})$  admits a (smooth) reduction  $(Q, M, G)$  and that  $G$  acts transitively, by symplectic diffeomorphisms, on a symplectic manifold  $(F, \varepsilon)$ .

Let  $E$  be the bundle with typical fibre  $F$  associated to  $Q$ , through the action of  $G$  on  $F$ . We shall, also, denote by  $\varepsilon$  the induced symplectic structure on the foliation formed by the fibres of  $E$ .

Let  $\mathcal{H} \subseteq TE$  be the connection on  $E$  induced by the Chern connection (see [4, p. 185]) of  $Q$ , determined by  $P$ . As  $\pi^{-1}(TM) = \mathcal{H}$ , where  $\pi : E \rightarrow M$  is the projection, the complex structure of  $M$  induces a decomposition  $\mathcal{H}^{\mathbb{C}} = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1}$ .

**Theorem 1.1.**  *$L(\mathcal{H}^{1,0} \oplus (\ker d\pi), i\varepsilon)$  is a generalized complex structure on  $E$ , where  $\varepsilon$  is extended, over  $\mathcal{H}$ , such that  $\iota_X \varepsilon = 0$ , for any  $X \in \mathcal{H}$ .*

*Proof.* Let  $\rho : G \rightarrow F$  be the projection determined by the action of  $G$ , by fixing a point of  $F$ . Then  $\eta = \rho^*(\varepsilon)$  is left-invariant and therefore determines a presymplectic structure, on the foliation formed by the fibres of  $Q$ , which we shall, also, denote by  $\eta$ .

Let  $\mathcal{K} \subseteq TQ$  be the Chern connection on  $Q$ , determined by  $P$ , and extend  $\eta$  over  $\mathcal{K}$  such that  $\iota_X \eta = 0$ , for any  $X \in \mathcal{K}$ . Also, the complex structure of  $P$  induces a decomposition  $\mathcal{K}^{\mathbb{C}} = \mathcal{K}^{1,0} \oplus \mathcal{K}^{0,1}$ .

Denote  $L_E = L(\mathcal{H}^{1,0} \oplus (\ker d\pi), i\varepsilon)$  and  $L_Q = L(\mathcal{K}^{1,0} \oplus (\ker d\pi_Q), i\eta)$ , where  $\pi_Q : Q \rightarrow M$  is the projection. By using [5, Proposition 1.3], it is easy to see that  $\rho^*(L_E) = L_Q$  and  $\rho_*(L_Q) = L_E$ , where  $\rho$ , also, denotes the projection from  $Q$  onto  $E$ . Consequently,  $L_E$  is integrable if and only if  $L_Q$  is integrable.

Now, the integrability of  $\mathcal{K}^{1,0} \oplus (\ker d\pi_Q)$  is an immediate consequence of the fact that the curvature form of a Chern connection is of type  $(1, 1)$ .

To prove that  $d\eta = 0$ , on  $\mathcal{K}^{1,0} \oplus (\ker d\pi_Q)$ , it is sufficient to consider two

cases. For this, let  $X, Y$  be basic sections of  $\mathcal{K}^{1,0}$  and let  $A, B$  be fundamental vector fields on  $Q$ . Then, as  $[X, A] = [X, B] = 0$  and  $\eta(A, B)$  is constant, we have  $d\eta(X, A, B) = X(\eta(A, B)) = 0$ . Also, by using, again, that the curvature form of  $\mathcal{K}$  is of type  $(1, 1)$ , we obtain  $d\eta(X, Y, A) = -\eta([X, Y], A) = 0$ .

The proof is complete.  $\square$

We have two classes of concrete examples for our construction:

(I)  $G = F$  is Abelian and endowed with an invariant symplectic structure. For example, if  $G = (S^1)^{2k}$  then  $P$  is the bundle of adapted frames of a holomorphic vector bundle of rank  $2k$ , over  $M$ , which splits into the direct sum of holomorphic line bundles. Consequently,  $Q$  is the bundle of adapted frames of some Hermitian structure on the corresponding holomorphic vector bundle. Moreover, if the Hermitian structure is real-analytic then the fibres of  $P$  endowed with  $\varepsilon^{\mathbb{C}}$  are the leaves of the symplectic foliation of the canonical (holomorphic) Poisson quotient (see [5, Theorem 2.3]) of the complexification of  $(E, L_E)$ .

(II)  $F$  is a coadjoint orbit endowed with the symplectic structure induced by the canonical Poisson structure on the Lie algebra of  $G$ . For example, the total space of any flag bundle of the tangent bundle of a Hermitian manifold is endowed, by Theorem 1.1, with a generalized complex structure.

With the same notations as above let  $\varphi : N \rightarrow M$  be a holomorphic map. Then, on endowing  $\varphi^{-1}(E)$  with the generalized complex structure induced by  $\varphi^{-1}(P)$  and  $\varphi^{-1}(Q)$ , the canonical bundle map  $\varphi^{-1}(E) \rightarrow E$  is holomorphic.

Similarly, any equivariant Poisson morphism from  $F$  to some homogeneous symplectic manifold determines a holomorphic map between generalized complex manifolds constructed as in Theorem 1.1.

We end by noting that case (I), above, can be strengthened, as follows.

**Corollary 1.2.** *Let  $G = (S^1)^{2k}$  and let  $(Q, M, G)$  be a principal bundle. Then there exists a natural bijective correspondence between the following:*

- (i) *Pairs formed of an invariant symplectic structure on  $G$  and a holomorphic structure on the extension of  $Q$  to  $G^{\mathbb{C}}$ ;*
- (ii)  *$G$ -invariant generalized complex structures, in normal form, on  $Q$  for which the foliations of the associated Poisson structures are given by the fibres of  $Q$ .*

*Proof.* It is sufficient to prove that any generalized complex structure  $L$  on  $Q$  satisfying (ii) is obtained as in Theorem 1.1.

Firstly, as  $L$  is  $G$ -invariant, its Poisson structure is, also,  $G$ -invariant. Together with the fact that  $L$  is integrable this implies that the Poisson structure of  $L$  is induced by some invariant symplectic structure on  $G$ .

Let  $\mathcal{D}$  be the co-CR structure determined by  $L$ . Then  $d\pi(\mathcal{D})$  is a complex structure on  $M$ , where  $\pi : Q \rightarrow M$  is the projection.

Also, if  $\mathcal{C}$  is the CR structure determined by  $L$  then  $\mathcal{K}^{\mathbb{C}} = \mathcal{C} \oplus \overline{\mathcal{C}}$ , where  $\mathcal{K}$  is a principal connection on  $Q$ . Moreover, the fact that  $\mathcal{C}$  is integrable is equivalent to the fact that the curvature form of  $\mathcal{K}$  is of type  $(1, 1)$ .

Thus, by a classical result, due to Koszul and Malgrange, there exists a unique holomorphic structure on the extension of  $Q$  to  $G^{\mathbb{C}}$  whose Chern connection is  $\mathcal{K}$ .  $\square$

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